Scientia Plena
www.scientiaplena.org.br
VOL. 19, NUM. 11 2023
doi: 10.14808/sci.plena.2023.119902

Perturbation methods for nonlinear eigenvalue problems

Métodos perturbativos para problemas de autovalores não lineares

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(Recebido em 22 de fevereiro de 2023; aceito em 14 de agosto de 2023)

In this paper, the dynamics of a microbeam is investigated from the point of view of nonlinear oscillations. Because it regards a non-linear problem, the natural frequency is more complex to obtain. Phenomena such as bifurcations and doubling periods, common in nonlinear systems, may appear. To carry out the ongoing analysis, two components are necessary: first, the equations of motion and, second, the techniques for investigating the behavior of the system. With respect to the equations of motion, deformation gradient theory is used. Concerning the second component, the following approach is employed: techniques of perturbation methods due to the non-linearities present in the model, with the objective of analyzing its oscillations. The important contribution of this investigation resides in a new approach to the equations of motion originated from the formulation of deformation gradient for the context of beams. For future research, it is intended to propose a new stiffness matrix. In the section of computational experiments, results that simulate the behavior of the eigenvalues, eigenfunctions, and solutions of the equation of motion are presented.
Keywords: deformation gradient, solid mechanics, perturbation methods.

Neste trabalho, a dinâmica de uma microviga e investigada do ponto de vista de oscilações não lineares. Por se tratar de um problema não linear, a frequência natural é mais complexa de se obter. Fenômenos como bifurcações e duplicação de períodos, comuns em sistemas não lineares, podem aparecer. Para se fazer a análise, são necessários dois componentes: primeiro, as equações de movimento e, segundo, as técnicas para investigar o comportamento do sistema. Com respeito as equações de movimento, a teoria de deformação gradiente e usada. Com respeito ao segundo componente, a seguinte abordagem será utilizada: técnicas de métodos perturbativos devido as não linearidades presentes no modelo, com o objetivo de analisar suas oscilações. A importante contribuição do presente trabalho reside em uma nova abordagem das equações de movimento originadas a partir da formulação da deformação gradiente para o contexto de vigas. Para trabalhos futuros, pretende-se propor uma nova matriz de rigidez. Na parte de experimentos computacionais, são apresentados resultados que simulam o comportamento dos autovalores, autofunções e soluções da equação de movimento.
Palavras-chave: deformação gradiente, mecânica dos sólidos, métodos perturbativos.

1. INTRODUCTION

Micromechanical structures play an important role in science and technology, according to Vatankhah et al. (2015) [1]. As an example of applications of such technologies, we can cite microforces used in the inspection and characterization of surfaces, according to Arjmand et al. (2008) [2]; microswitches used to control high frequencies, according to Joglekar and Pawaskar (2011) [3] and microresonators acting at specific frequencies, study carried out by Hassanpour et al. (2010) [4]. According to Seok and Scarton (2006) [5], due to its efficiency, simplicity in structural architecture and production, all this equipment, along with a wide variety of applications such as micropressurizers, mass sensors and microflexible joints make them an attractive audience in the micro equipment production industry, according to studies by Vatankhah et al. (2015) [1].

One of the great difficulties in the analysis of oscillations in microstructures refers to the equations of motion of the system. Recent studies presented the motion equations of electrostatic actuators, allowing the identification of oscillations of such mechanisms subject to specific boundary conditions based on the theory of deformation gradient, as per Qian et al. (2012) [6]. Also, properties of *pull-in* instability and *pull-in* tension were systematically investigated by several authors, like Mojahedi et al. (2010) [7], for instance.

The growing need for better and more accurate performance of microscale instruments has increased research in this area, particularly with regard to modeling issues. As an example of problems raised in the literature lies in the work of Vagia (2012) [8], who studied the design of linear and non-linear controllers and actuators for the suppression of vibration modes of microstructures. Furthermore, Wang (1998) [9] covered the feedback control switched for vibration stabilization, and Yen et al. (2005) [10] proposed the use of sliders discrete to eliminate vibrations caused by moving bases.

In the qualitative analysis of microstructures, such as movement stability, vibration stabilization of controllers and actuators, one must take into account the set of parameters that interfere with both the dynamics and the analysis of the performance of such systems. Therefore, the development of order reduction techniques on the parameter set is important. In this sense, Vagia et al. (2008) [11] present order reduction techniques based on the Rayleigh-Ritz method.

In recent papers, Vatankhah et al. (2013) [12] proposed a new deformation model considering terms of order three in the curvature expression, introducing non-classical continuum mechanics, called deformation gradient theory by some authors, according to Vatankhah et al. (2013) [13]. In the context of the theory of elasticity, expressive contributions were presented by Lam et al. (2003) [14]. Using deformation gradient theory, new equations of motion were presented to better describe the displacement of microbeams. Among the many contributions, we mention Vatankhah et al. (2014) [15], who studied dynamic behavior with static characteristics, also called bifurcation theory, dynamic behavior, and stability properties of microbeams; deformations by static deflection, buckling and free vibrations of microbeams, according to Ansari et al. (2013) [16]; forced system vibrations, as per Vatankhah et al. (2013) [13].

All things considered; the objective of this paper is to analyze the behavior of the microbeam. For this, it is assumed that the displacement of the structure obeys a harmonic type movement. Due to the non-linearity of the electric field and the considerable amount of parameters present in the model, the present methodology naturally leads to renormalization problems, which allows the introduction of perturbation methods in the analysis. Such methodology makes possible the calculation of the eigenvalues of the system providing a dynamic analysis of the behavior of the structure. Finally, a case study is done to compare the results obtained with values already used in the literature. In this sense, a new stiffness matrix for the deflection of beams based on the theory of deformation gradient is proposed.

2. MATERIALS AND METHODS

The system considered is a microbeam fixed at one end and free at the other, which is subjected to a deflection force F, which is specified by the electrodes, as seen in Figure 1.

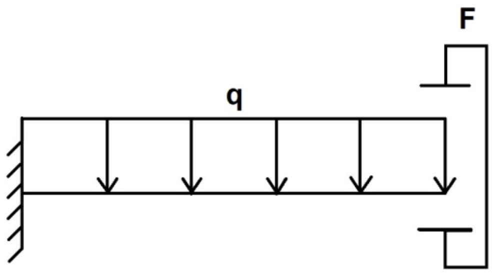

Figure 1: Beam control scheme.

When the electric field (charge q) is activated, the control of the device is triggered and the equilibrium of the system is sought from the following parameters: constant cross-sectional area, denoted by A, density ρ, length L thickness h. The partial differential equation that models the beam displacement as well as the boundary conditions are obtained using Hamilton's principle along with deformation gradient theory.

According to Lam et al. (2003) [14] deformation gradient theory, the equation of motion of the microbeam that describes the behavior of w, with constant cross section through the axis along its length, subject to excitations and control parameters described in Figure 1, is given by the following equation:

$$\rho A \partial_{tt} w + k_1 \partial_x^4 w - k_2 \partial_x^6 w = F(w,d), x \in [0,L], t \geq 0, \tag{1}$$

It is also subject to the following boundary conditions:

$$\begin{aligned} w(0,t) &= 0, \\ \partial_x w(0,t) &= 0, \\ \partial_x^2 w(0,t) &= 0, \\ \partial_x^3 w(0,t) &= 0, \\ \partial_x^5 w(0,t) &= 0, \\ \partial_x^4 w(L,t) - \partial_x^2 w(L,t) &= 0, t \geq 0. \end{aligned} \tag{2}$$

For more details on the physical meaning of boundary conditions, see Huston and Josephs (2008) [17]. The parameters $k_i, i = 1,2$ are given, respectively, by

$$k_1 = EI + \mu A \left(2l_0^2 + \frac{43}{225} l_1^2 + l_2^2\right) \tag{3}$$

$$k_2 = \mu I \left(2l_0^2 + \frac{4}{5} l_1^2\right), \tag{4}$$

where

- l is the second moment of area of the beam;
- E is Young's modulus;
- μ is the shear modulus;
- $l_i, i = 0,1,2$, represent the parameters associated with gradient dilation (l_0); the divergent resistance (l_1) and the gradient rotation (l_2).

For more details, see Vatankhah et al. (2013) [12].

The electrical forces acting on the beam to maintain its equilibrium when subjected to external forces are given by

$$F(x,w,d) = \frac{1}{2} \epsilon \psi(x)((d-w)^{-2} - (d+w)^{-2}), \tag{5}$$

where

- ϵ corresponds to the electrical allowability;
- d refers to the distance between the electrodes;
- w is the beam displacement vector.

3. RESULTS AND DISCUSSION

This section aims to determine the frequency equations for the problem. For this, consider the following dimensionless group

$$\tau = \frac{t}{T}, \xi = \frac{x}{\delta L}, w = dv. \tag{6}$$

By replacing (6) to (1), we get:

$$\frac{\rho A}{T^2} \partial_{tt} v(\tau,\xi) + \frac{k_1}{\delta^4 L^4} \partial_\xi^4 v(\tau,\xi) - \frac{k_2}{\delta^6 L^6} \partial_\xi^6 v(\tau,\xi) =$$

$$= -0.5\epsilon\psi(\xi)\left(\left(d - v(\tau,\xi)\right)^{-2} - \left(d + v(\tau,\xi)\right)^{-2}\right), \tag{7}$$

where w and v are related by the following equation

$$w(t,x) = v(\tau,\xi). \tag{8}$$

The parameters T and δ are defined as

$$T = \frac{\sqrt{\rho A k_2}}{k_1\sqrt{k_1}}, \tag{9}$$

$$\delta = \frac{\sqrt{k_2}}{\sqrt{k_1}L}. \tag{10}$$

On the other hand, the perturbation parameter ε is defined as

$$\varepsilon = \frac{0.5\epsilon k_2^2}{d^3 k_1^3}. \tag{11}$$

It will be assumed that the solution is separable into x and t, which means that the solution (1) is given by

$$w(x,t) = w(x)e^{i\omega t}. \tag{12}$$

By introducing (12) into (1) and considering stationary motion, we obtain

$$\begin{gathered} k_1 w^{iv} - k_2 w^{vi} + \lambda\rho A w = 0.5\epsilon\psi(x)((d-w)^{-2} + (d + w)^{-2}), \\ Bi(w) = 0, \end{gathered} \tag{13}$$

where $\lambda = -\omega^2$ and $BI, i = 1{:}\,6$ are the boundary conditions given in (1).

In the analysis of the eigenvalue problem (13), some definitions referring to functional analysis are necessary. First, we will assume that the functions of interest are real and differentiable in the domain $D{:}\ 0 < x < L$. The solution space of the problem, to be correctly defined, must consider the operator acting on (13), which is of the order of $6 = 2p, p = 3$. This space will be denoted by $\|_B^{2p}$ showing that the derivatives of order $2p$ of w have finite energy, which is

$$\int_0^L w^2 dx < \infty, \tag{14}$$

with the boundary conditions (1). It follows from (7) that the equation (15) is valid

$$Hw + \lambda w = \varepsilon F(w), \tag{15}$$

subject to boundary condition

$$B(w) = 0, \tag{16}$$

where H is the operator given by

$$H = \frac{d^4}{dx^4} - \frac{d^6}{dx^6}. \tag{17}$$

$B(\cdot)$ is the boundary conditions given in (1) and $F\,(\cdot)$ is the non-linear function given by

$$F(w) = 0.5\psi(x)((1 - w)^{-2} - (1 + w)^{-2}). \tag{18}$$

For small ε, we have looked for a solution given by

$$\begin{aligned} w &= w_0 + \varepsilon w_1 + \cdots, \\ \lambda &= \lambda_0 + \varepsilon \lambda_1 + \cdots. \end{aligned} \tag{19}$$

By substituting (19) into (15) and equating the terms of the same power, we propose the following boundary value problem

$$\begin{aligned} &Hw_0 + \lambda_0 w_0 = 0,\ B(w_0) = 0, \\ &Hw_1 + \lambda_0 w_1 = \lambda_1 w_0 + F(w_0). \end{aligned} \tag{20}$$

The first equation of (16) can be written as

$$\begin{aligned} &\frac{d^4 w_0}{dx^4} - \frac{d^6 w_0}{dx^6} + \lambda_0 w_0 = 0 \\ &B(w_0) = 0, \end{aligned} \tag{21}$$

Assuming that $w_0 = e^{sx}$ and substituting in (21) we obtain the frequency equation

$$s^4 - s^6 + \lambda = 0. \tag{22}$$

The general solution of (21) is given by

$$\emptyset = \sum_{i=1}^{6} c_i e^{si(\lambda)x} \tag{23}$$

where $c_i, i = 1{:}\,6$ are arbitrary constants. To determine the frequency equation for λ we consider the Wronskian

$$wr_{ij}(x) = s_i^j e^{s_i x},\ i = 1 : 6,\ j = 0 : 5. \tag{24}$$

By using the boundary conditions of the problem, we achieve the following matrix

$$wr(0, L, \lambda) = \\ = \left| \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_1^2 & s_2^2 & s_3^2 & s_4^2 & s_5^2 & s_6^2 \\ s_1^3 e^{s_1 L} & s_2^3 e^{s_2 L} & s_3^3 e^{s_3 L} & s_4^3 e^{s_4 L} & s_5^3 e^{s_5 L} & s_6^3 e^{6L} \\ (s_1^4 - s_1^2)e^{s_1 L} & (s_2^4 - s_2^2)e^{s_2 L} & (s_3^4 - s_3^2)e^{s_3 L} & (s_4^4 - s_4^2)e^{s_4 L} & (s_5^4 - s_5^2)e^{s_5 L} & (s_6^4 - s_6^2)e^{s_6 L} \\ s_1^5 & s_2^5 & s_3^5 & s_4^5 & s_5^5 & s_6^5 \end{pmatrix} \right|. \tag{25}$$

For the purpose of numerical experiments, it is shown that

$$wr(0, L, \lambda) = \prod_{i=1}^{6} \sigma_i, \tag{26}$$

where $\sigma_i(\lambda), i = 1{:}\,6$ are the eigenvalues of $wr(0, L, \lambda)$. By using the variational principle, $\sigma_i(\lambda)$ can be written as

$$\sigma_i(\lambda) = \frac{u_i^{\perp} Wr(0, L, \lambda) u_i}{u_i^{\perp} u_i}, \tag{27}$$

where u_i is the eigenvector of $wr(0, L, \lambda)$ associated with the eigenvalue σ_i. The previous considerations now allow us to determine the frequency equation for (21)

$$wr(0, L, \lambda) = 0. \tag{28}$$

For greater precision of the behavior of the eigenfunctions, Figure 2 depicts the imaginary part of the roots of (28) for a range of values for the eigenvalues λ between -1.5 and 1.5. It appears that, depending on the initial conditions of the problem, the eigenfunctions may have a minimum or maximum frequency, which shows the oscillatory character of the roots of (28).

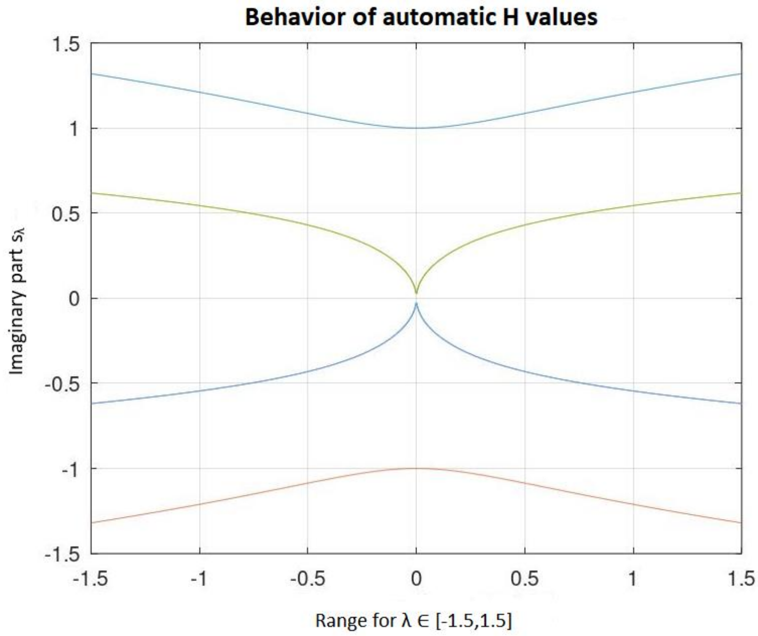

Figure 2: Behavior imaginary part roots of the characteristic equation of H.

Figure 3 presents the behavior of the real part of the roots of the characteristic equation (22).

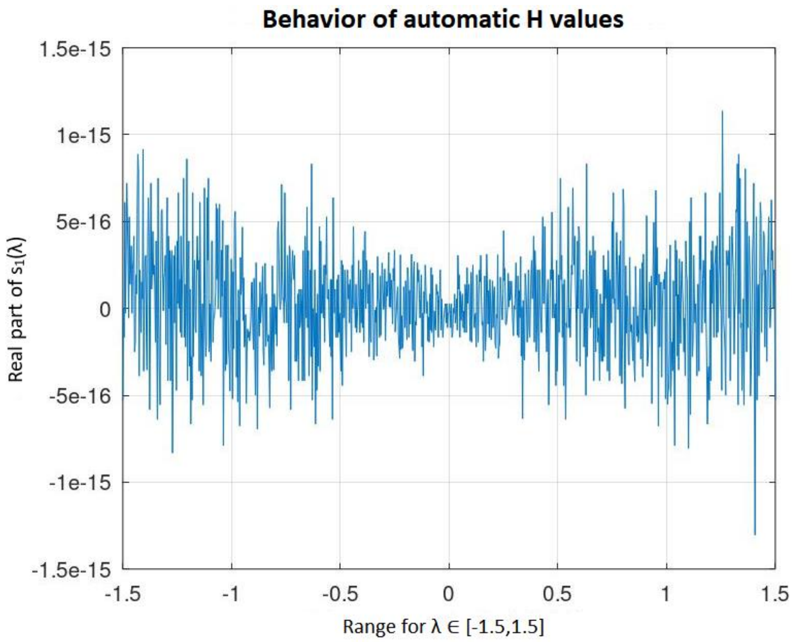

Figure 3: Behavior real part roots of the characteristic equation of H.

At this point, it is important to observe in Figure 3 that the range of oscillation amplitude of the roots of the equation (22) is in the range of $O(-16)$, characterizing the stability of the solution. Finally, Figures 4 and 5 show the behavior of both the real and imaginary parts of the eigenfunctions.

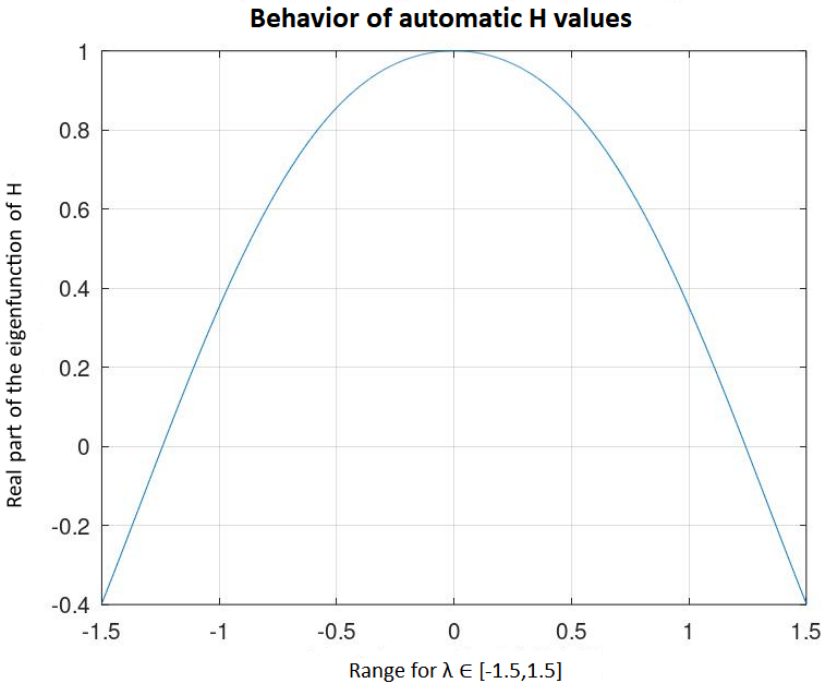

Figure 4: Behavior imaginary part roots of the characteristic equation of H.

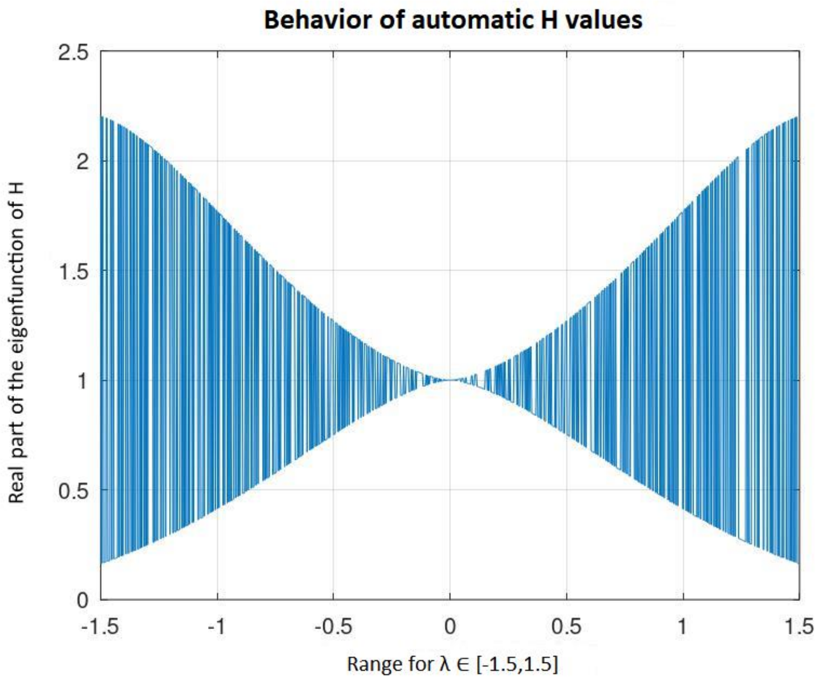

Figure 5: Behavior part real eigenfunctions of H.

By considering the expansion (19), we have

$$H\big(\phi_0 + \epsilon\phi_1 + g(\epsilon^2)\big) + \lambda\big(\phi_0 + \epsilon\phi_1 + g(\epsilon^2)\big) = \epsilon\big(\phi_0 + \epsilon\phi_1 + g(\epsilon^2)\big). \quad (29)$$

By using the linearity of H , the expression at (29) becomes

$$H\phi_0 + \lambda_0\phi_0 + \epsilon(H\phi_1 + \lambda_1\phi_0 + +\lambda_0\phi_1) + g(\epsilon^2) = \epsilon F\big(\phi_0 + \epsilon\phi_1 + g(\epsilon^2)\big). \quad (30)$$

By using the expansion of F,

$$\begin{aligned} F &= 0.5\psi(x)((1-x)^{-2} - (1+x)^{-2}) \\ &= 0.5\psi(x)((1 + 2x + g(\epsilon^2)) - (1 - 2x + g(\epsilon^2))) \\ &= 0.5\psi(x)(4x + g(\epsilon^2)), \end{aligned} \quad (31)$$

the expression in (30) becomes

$$\begin{aligned} F(\phi) &= 0.5\varepsilon\psi(x) * \big(4\phi + g(\phi^2)\big) \\ &= 2\varepsilon\psi(x)\phi_0 + \varepsilon 2\epsilon\psi(x)\phi_1 g(\epsilon^2) \end{aligned} \quad (32)$$

By returning again in the expression (30) and equating the terms of the same power,

$$\begin{aligned} H\phi_0 + \lambda_0\phi_0 = 0, B(\phi_0) = 0, \\ H\phi_1 + \lambda_0\phi_1 = -\lambda_1\phi_0 + 2\varepsilon\psi(x)\phi_0. \end{aligned} \quad (33)$$

we will consider ϕ_1 as the direct sum of the eigenfunctions um of H. We will also put $\phi_{0n} = u_n$ and $\lambda_{0n} = \mu_n$. By using the orthogonality of the eigenfunctions un, problem (33) can be written as

$$a_s(\mu_s - \mu_n) = -\lambda_1\delta_{sn} + 2\varepsilon\int_0^L \psi(x)u_n\bar{u}_s ds. \quad (34)$$

From the equation (34), we see that if $s = n$,

$$\lambda_{1n} = 2\varepsilon\int_0^L \psi(s)u_n^2 ds. \quad (35)$$

On the other hand, if $s \neq n$,

$$a_{ms} = \frac{2\varepsilon}{\mu_s - \mu_n}\int_0^L \psi(x)u_m u_s dx. \quad (36)$$

Is it possible to show $a_{mm} = 0$ in the analysis from Nayfeh (2008) [18]. This leads to the following formula for the eigenvalues of H,

$$\lambda_n = \lambda_{0n} + \varepsilon\lambda_{1n} + g(\varepsilon^2). \quad (37)$$

With the eigenvalues λ_n provided by the equation (37), we can now calculate the natural frequency of the beam which is given by the equation

$$w_n = -i\sqrt{\lambda_n}. \quad (38)$$

Finally, by using the information about the natural frequencies of the structure, the equation of state of the system becomes the following:

$$w_{n,i}(x,t) = e^{s_i^n x - jw_n t}. \quad (39)$$

4. CONCLUSION

In the present paper, the dynamic behavior of a microbeam subjected to an electric field has been studied. The analysis has been carried out using the theory of deformation gradient, which led to a characteristic equation of order six, in addition to the linearities inherent to the model. To obtain the eigenfunctions of the *H* operator, the perturbative method has been used. Numerical simulations have been analyzed in two ranges, [-1.50, 1.50] and [0.45, 1.50]. In the first range, a well-determined behavior of the imaginary part of the eigenvalues may be observed. With respect to the second range, attention is drawn to the oscillation amplitude of the real part of the eigenvalues varying in an interval around 10^{-15}. From the previous graphs, it has been observed that the system has several oscillations, converges between the two ranges mentioned above, reaching equilibrium. Thus, it is expected that the structure also have this stability behavior.

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